

Generalized Self-similar Scalar-Tensor Theories

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Abstract

We study through symmetry principles the form of the functions in the generalized scalar-tensor theories under the self-similar hypothesis. The results obtained are absolutely general and valid for all the Bianchi models and the flat FRW one. We study the concrete example of the Kantowsky-Sach model finding some new exact self-similar solutions.

1 Introduction

Current observations of the large scale cosmic microwave background suggest to us that our physical universe is expanding in an accelerated way, isotropic and homogeneous models with a positive cosmological constant. The analysis of Cosmic Microwave Background (CMB) fluctuations could confirm this picture. But other analyses reveal some inconsistencies. Analysis of WMAP data sets shows us that the universe might have a preferred direction. For this reason, it may be interesting to study Bianchi models since these models may describe such anisotropies.

The observed location of the first acoustic peak of the temperature fluctuations on the CMB corroborated by the data obtained in different experiments [1], indicates that the universe is dominated by an unidentified “dark energy” and suggests that this unidentified dark energy has a negative pressure [2]. This last characteristic of the dark energy points to the vacuum energy or cosmological constant as a possible candidate for dark energy. Although, it is a general belief that the current curvature of the universe is negligible and mostly the universe is considered with a flat geometry, recent observations support the possibility of a non-flat universe and detect a small deviation from $k = 0$ [3]. For example, evidence from CMB and also supernova measurements of the cubic correction to the luminosity distance favour a positively curved universe [4, 5].

In order to explain the current acceleration of the universe, within General Relativity (GR), it is necessary to introduce a new type of energy with a negative pressure. Between the different possible approaches is one which consists of considering so-called dark energy (DE). There are several candidates for DE, where the simplest one is the cosmological term Λ . However, this choice has several drawbacks such as coincidence and fine tuning problems. For this reason other models have been proposed. Examples of such models are quintessence [6, 7], K-essence [8], or chameleonic fields in which a scalar field is coupled to matter [9], etc. Hence it is natural and important to consider a variable Λ -term in more general frameworks where, furthermore, other quantities, such as the Newton gravitational constant may be considered as dynamical. One such class of theories are the scalar-tensor theories (STT) of gravity. This class of models has received a renewed interest in recent times, for two main reasons: First, the new inflationary scenario as the extended inflation has a scalar field that solves several problems present in the old theories. Secondly, string theories and other unified theories contain a scalar field which plays a similar role to the scalar field of the STT. The scalar-tensor theories started with the work of P. Jordan in 1950 [10]. A prototype of such models was proposed by Brans and Dicke in 1961 [11]. Their aim for presenting this model was to modify

Einstein's theory in such a way as to incorporate the so called "Mach's principle". These theories have been generalized by P.G. Bergmann [12], K. Nordtverdt [13] and R. T. Wagoner [14]. For a recent review of this class of theories we refer to [15] and [16].

In this paper we want to consider a family of scalar-tensor theories with a dynamical cosmological constant [17] and with a potential [15], that is equivalent to a time dependent cosmological constant. Recently several authors have considered the cosmological consequences of a time varying cosmological constant. Most of them introduce the time dependence in an ad hoc manner. In this work we consider an equivalent problem in the well known general scalar-tensor theory of gravity where the time dependence can occur in a natural way, without any new assumption or modification of the theory and provide an explanation for the acceleration of the universe expansion [18].

Many authors have studied these general scalar-tensor theories. They use the observational data in order to obtain restrictions or constraint between the functions that appear in the action to obtain an accelerated model [19]-[22]. Our approach is different: We want to derive these functions from symmetry principles as self-similarity. We shall carry out our study under this assumption and state some general theorems that are valid for all the Bianchi models and of course for the flat FRW one. It is most appropriate for us to work in the Jordan frame (JF), in which the physical quantities are those that are being measured in experiments, even though the Einstein frame (EF) often provides a better mathematical insight.

The study of self-similar (SS) models is quite important since a large class of orthogonal spatially homogeneous models are asymptotically self-similar at the initial singularity and are approximated by exact perfect fluid or vacuum self-similar power-law models. Exact self-similar power-law models can also approximate general Bianchi models at intermediate stages of their evolution. This last point is of particular importance in relating Bianchi models to the real Universe. At the same time, self-similar solutions can describe the behaviour of Bianchi models at late times i.e. as $t \rightarrow \infty$ [23].

This paper is organized as follows. In section two we start by considering a particular formulation of the theory. In this case the cosmological constant is introduced directly by the function $\Lambda(\phi)$ [17]. We study this model through two different approaches. The first consists of studying the effective stress-energy tensor under the matter collineation approach. This method allows us to obtain relationships between the physical quantities as well as to determine the exact form of the scalar field ϕ . The second approach consists of studying the wave equation under the Lie group method. By imposing a particular symmetry we are able to determine the exact form for each of the functions that appear in this equation. We summarize all the results by stating a very general theorem. In section three we study a very general scalar-tensor theory. In this case the cosmological constant is introduced by the potential. Following the same exposed procedure as in the above section, we are able to determine the exact form that all of the unknowns involved in this model must have. We show, from the stated theorem, how different versions of this theory arise, which are the standard Brans-Dicke theory, the induced gravity model [24] and a very particular solution where the effective gravitational function is constant. In section four we study a chameleon Jordan-Brans-Dicke model. In order to show how all the obtained results work, in section five we study a particular example which is the Kantowski-Sach model. We start this section by showing that this metric admits a homothetic vector fields and then we study several models. We put special emphasis on comparing the solutions obtained in each case. In section six we end by summarizing all the results. We have added an appendix where we study in detail one of the equations obtained in section 2 in order to show with out any doubt that the Brans-Dicke parameter $\omega(\phi)$ must be constant in this framework of self-similar solutions.

2 Cosmological models with dynamical Λ in scalar-tensor theories

Following to Will (see [17]) we start with the action for the most general scalar-tensor theory of gravitation

$$S = \frac{c^3}{16\pi G_N} \int d^4x \sqrt{-g} \left[\phi R - \frac{\omega(\phi) g^{ij} \phi_{,i} \phi_{,j}}{\phi} + 2\phi \Lambda(\phi) \right] + S_{NG}, \quad (1)$$

where $g = \det(g_{ij})$, G_* is Newton's constant, S_{NG} is the action for the nongravitational matter. We use the signature $(-, +, +, +)$. The arbitrary functions $\omega(\phi)$ and $\Lambda(\phi)$ distinguish the different scalar-tensor theories of gravitation, $\Lambda(\phi)$ is a potential function and plays the role of a cosmological constant, and $\omega(\phi)$ is the coupling function of the particular theory.

The explicit field equations are

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi}{c^4\phi}T_{ij} + \Lambda(\phi)g_{ij} + \frac{\omega}{\phi^2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,l}\phi^{,l}\right) + \frac{1}{\phi}\left(\phi_{;ij} - g_{ij}\square\phi\right), \quad (2)$$

$$(3 + 2\omega(\phi))\square\phi = 8\pi T - \frac{d\omega}{d\phi}\phi_{,l}\phi^{,l} - 2\phi\left(\phi\frac{d\Lambda}{d\phi} - \Lambda(\phi)\right), \quad (3)$$

where $T = T_i^i$ is the trace of the stress-energy tensor. The gravitational coupling $G_{\text{eff}}(t)$ is given by

$$G_{\text{eff}}(t) = \left(\frac{2\omega + 4}{2\omega + 3}\right) \frac{G_*}{\phi(t)}. \quad (4)$$

2.1 Matter collineations

We may calculate the relationship between the quantities (in a SS approach) by calculating the matter collineations. Therefore we have to compute (we use unit where $8\pi = c = 1$)

$$T_{ij}^{\text{eff}} = \frac{1}{\phi}T_{ij} + \frac{\omega(\phi)}{\phi^2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,l}\phi^{,l}\right) + \frac{1}{\phi}\left(\phi_{;ij} - g_{ij}\square\phi\right) + \Lambda(\phi)g_{ij}, \quad (5)$$

$$\mathcal{L}_{HO}(T_{ij}^{\text{eff}}) = 0, \quad (6)$$

where HO stands for a homothetic vector field. For simplicity we have used a flat FRW metric but we would like to emphasize that all the obtained results are absolutely valid for all the Bianchi models, since we only look for the behaviour of the physical quantities instead of restriction on the scale factors. Therefore the homothetic vector field (HVF) is

$$HO = (t + t_0)\partial_t + (1 - (t + t_0)H)x\partial_x + (1 - (t + t_0)H)y\partial_y + (1 - (t + t_0)H)z\partial_z. \quad (7)$$

Note the non-singular character of the HVF, nevertheless for simplicity in the calculations we use the singular case.

$$1. \quad T_1 = \phi^{-1}T_{ij},$$

$$\mathcal{L}_{HO}\left(\frac{1}{\phi}T_{ij}\right) = 0, \quad \Longleftrightarrow \quad -t\rho\phi' + t\rho'\phi + 2\rho\phi = 0, \quad (8)$$

obtaining

$$\frac{\rho'}{\rho} - \frac{\phi'}{\phi} = -\frac{2}{t} \quad \Longleftrightarrow \quad \frac{\rho}{\phi} = t^{-2}. \quad (9)$$

$$2. \quad T_2 = \omega(\phi)\phi^{-2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,l}\phi^{,l}\right)$$

$$\mathcal{L}_{HO}\left(\frac{\omega(\phi)}{\phi^2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,l}\phi^{,l}\right)\right) = 0 \quad \Longleftrightarrow \quad t\phi_t^2(\omega_\phi\phi - 2\omega) + 2t\omega\phi_{tt}\phi + 2\omega\phi_t\phi = 0, \quad (10)$$

and therefore

$$\phi_{tt} = -\frac{\phi_t^2}{\phi}\left(\frac{\omega_\phi\phi}{2\omega} - 1\right) - \frac{\phi_t}{t}, \quad (11)$$

or

$$\frac{\omega_\phi}{\omega}\phi_t - 2\frac{\phi_t}{\phi} + 2\frac{\phi_{tt}}{\phi_t} = -\frac{2}{t} \quad \frac{\omega'}{\omega} - 2\frac{\phi_t}{\phi} + 2\frac{\phi_{tt}}{\phi_t} = -\frac{2}{t} \quad \Longleftrightarrow \quad \omega(\phi)\frac{\phi_t^2}{\phi^2} = t^{-2}. \quad (12)$$

note that $\omega' = \omega_\phi \phi_t$.

We find the next solution for Eq. (11)

$$\int^\phi \frac{\sqrt{\omega(\phi)}}{\phi} d\phi - C_1 \ln t + C_2 = 0. \quad (13)$$

For example

- $\omega(\phi) = \text{const}$

$$\int^\phi \frac{k}{\phi} d\phi = k \ln \phi, \quad \implies \quad \phi = \phi_0 t^n, \quad (14)$$

this is the unique solution mathematically possible (compatible with the SS hypothesis, see the appendix for an explanation).

- $\omega(\phi) = \phi^a$

$$\int^\phi \frac{\phi^{a/2}}{\phi} d\phi = \frac{2}{a} \phi^{a/2} \quad \implies \quad \frac{2}{a} \phi^{a/2} = C_1 \ln t \quad \phi = C_1 (\ln t)^{2/a}. \quad (15)$$

If $\omega = \text{const}$. then

$$t\phi_t^2 - t\phi_{tt}\phi - \phi_t\phi = 0, \quad (16)$$

and therefore we get the following ODE

$$\phi_{tt} = \frac{\phi_t^2}{\phi} - \frac{\phi_t}{t} \quad \iff \quad \phi = \exp(-C_3 - C_2 \ln t) = \phi_0 t^{C_2}. \quad (17)$$

$$3. \quad T_3 = \phi^{-1} (\phi_{;ij} - g_{ij} \square \phi)$$

$$\mathcal{L}_{HO} \left(\frac{1}{\phi} (\phi_{;ij} - g_{ij} \square \phi) \right) = 0, \quad (18)$$

i.e.

$$t \left(\phi_{tt} f' + \phi_t f'' - \frac{\phi_t^2}{\phi} f' - \phi_t \frac{f'^2}{f} \right) + 2\phi_t f' = 0, \quad (19)$$

$$t \left[\phi''' + \left(2H - \frac{\phi'}{\phi} \right) \phi'' - 2H \frac{\phi'^2}{\phi} + 2H' \phi' \right] + 2(\phi'' - 2\phi' H) = 0, \quad (20)$$

note that $H = ht^{-1}$, $h \in \mathbb{R}^+$. These equation are different for each Bianchi model and we only obtain restriction on the scale factors.

$$4. \quad T_4 = \Lambda(\phi) g_{ij},$$

$$\mathcal{L}_{HO} (\Lambda(\phi) g_{ij}) = 0 \quad (21)$$

i.e.

$$t\Lambda_\phi \phi' + 2\Lambda = 0 \quad \iff \quad \frac{\Lambda_\phi}{\Lambda} \phi' = -\frac{2}{t} \quad \iff \quad \Lambda = \Lambda_0 t^{-2}, \quad (22)$$

where $\Lambda' = \Lambda_\phi \phi'$.

2.2 Lie groups

We are going to study the Eq. (3) through the LG method, i.e. we study the kind of functions $\Lambda(\phi)$ and $\omega(\phi)$ such that this equation is integrable. We start by rewriting it in an appropriate way

$$(3 + 2\omega(\phi)) (\phi'' + ht^{-1} \phi') = Ct^{-\alpha} + B(\Lambda - \phi \Lambda_\phi) \phi - \phi'^2 \omega_\phi, \quad (23)$$

where $h = \text{const.}$, $h \in \mathbb{R}^+$, $B = 2$, and $C = 8\pi(1 - 3\gamma)\rho_0$. Note that we are taking into account the conservation equation $\text{div } T = 0$, i.e. $\rho = \rho_0 t^{-\alpha}$, where $\alpha = (1 + \gamma)h$, and $H = ht^{-1}$.

We need to solve the following system of PDE

$$\omega_\phi \xi_\phi - W \xi_{\phi\phi} = 0, \quad (24)$$

$$2ht^{-1}W \xi_\phi + W \eta_{\phi\phi} - 2W \xi_{\phi t} - (2W^{-1}\omega_\phi^2 - \omega_{\phi\phi}) \eta + \omega_\phi \eta_\phi = 0, \quad (25)$$

$$\begin{aligned} & -3(B\phi(\Lambda - \phi\Lambda_\phi) + Ct^{-\alpha}) \xi_\phi + ht^{-2}W(t\xi_t - \xi) + 2W\eta_{t\phi} - \\ & - W\xi_{tt} + 2ht^{-1}\omega_\phi(1 - (3 + 2\omega)W^{-1}) \eta + 2\omega_\phi \eta_t = 0, \end{aligned} \quad (26)$$

$$\begin{aligned} & [B(\phi^2\Lambda_{\phi\phi} + \phi\Lambda_\phi - \Lambda) + 2\omega_\phi W^{-1}(B\phi(\Lambda - \phi\Lambda_\phi) + Ct^{-\alpha})] \eta - 2[B\phi(\Lambda - \phi\Lambda_\phi) - Ct^{-\alpha}] \xi_t + \\ & + \alpha Ct^{-\alpha-1} \xi + [B\phi(\Lambda - \phi\Lambda_\phi) + Ct^{-\alpha}] \eta_\phi + ht^{-1}W\eta_t + W\eta_{tt} = 0, \end{aligned} \quad (27)$$

where $W = (3 + 2\omega(\phi))$. notice that $(2\omega + 3)W^{-1} = 1$, so Eq. (26) yields

$$-3(B\phi(\Lambda - \phi\Lambda_\phi) + Ct^{-\alpha}) \xi_\phi + ht^{-2}W(t\xi_t - \xi) + 2W\eta_{t\phi} - W\xi_{tt} + 2\omega_\phi \eta_t = 0. \quad (28)$$

The symmetry $\xi = t$, $\eta = n\phi$, brings us to obtain the following restriction on $\Lambda(\phi)$. From Eq. (27) we get

$$\begin{aligned} & [B(\phi^2\Lambda_{\phi\phi} + \phi\Lambda_\phi - \Lambda) + 2\omega_\phi W^{-1}(B\phi(\Lambda - \phi\Lambda_\phi) + Ct^{-\alpha})] n\phi - 2[B\phi(\Lambda - \phi\Lambda_\phi) - Ct^{-\alpha}] + \\ & + \alpha Ct^{-\alpha} + [B\phi(\Lambda - \phi\Lambda_\phi) + Ct^{-\alpha}] n = 0, \end{aligned} \quad (29)$$

so

$$Ct^{-\alpha}(\alpha + n - 2 + 2n\omega_\phi W^{-1}) = 0, \quad (30)$$

and

$$n(\phi^2\Lambda_{\phi\phi} + \phi\Lambda_\phi - \Lambda) + (\Lambda - \phi\Lambda_\phi)(-\alpha) = 0, \quad (31)$$

where we have taken into account Eq. (30) therefore we have obtained the following ODE for $\Lambda(\phi)$

$$\Lambda_{\phi\phi} = \left(\frac{n + \alpha}{n}\right) \left(-\frac{\Lambda_\phi}{\phi} + \frac{\Lambda}{\phi^2}\right), \quad (32)$$

and whose general solution is

$$\Lambda(\phi) = \Lambda_0 \phi^{\frac{-1}{n}(n+\alpha)} + \frac{C_3 n}{2n + \alpha} \phi, \quad (33)$$

we choose

$$\Lambda(\phi) = \Lambda_0 \phi^{\frac{-1}{n}(n+\alpha)} = \Lambda_0 t^{-(n+\alpha)}, \quad (34)$$

since from our result from the MC approach we already know that, $\Lambda(t) = t^{-2}$, so this means that $n + \alpha = 2$.

In the same way we may calculate the restriction on $\omega(\phi)$,

$$\omega_\phi \phi W^{-1} = \frac{2 - \alpha - n}{2n}, \quad \omega' = \left(\frac{2 - \alpha - n}{2n}\right) \frac{(3 + 2\omega)}{\phi}, \quad (35)$$

whose solutions are

$$\omega(\phi) = \phi^{\frac{-1}{n}(n+\alpha-2)} e^{-2nC_4} - \frac{3}{2} = \omega_n t^{-(n+\alpha-2)} - \frac{3}{2}, \quad \omega_n = \text{const}, \quad (36)$$

therefore we obtain

$$\omega(\phi) = \text{const.}, \quad n + \alpha = 2. \quad (37)$$

In an alternative way, from Eq. (25) we get

$$\omega_{\phi\phi} = \left(\frac{2 - \alpha - 2n}{n}\right) \frac{\omega_\phi}{\phi}, \quad \implies \quad \omega(\phi) = C_8 + C_9 \phi^{\frac{1}{n}(2-n-\alpha)} = \omega_0 t^{2-n-\alpha}, \quad (38)$$

and therefore

$$\omega(\phi) = \omega_0 = \text{const}, \quad (39)$$

since $2 = n + \alpha$. In the appendix we shall give an alternative and detailed proof of this result.

Theorem 1 *The scaling and in particular the self-similar solution admitted for the FE (2-3) have the following form*

$$\phi = \phi_0 (t + t_0)^n, \quad \Lambda(\phi) = \Lambda_0 \phi^{\frac{-1}{n}(n+\alpha)} = \Lambda_0 (t + t_0)^{-(n+\alpha)},$$

with $n + \alpha = 2$, therefore $\Lambda(t) = \Lambda_0 (t + t_0)^{-2}$. The Brans-Dicke parameter is constant

$$\omega(\phi) = \text{const.}$$

and $\rho = \rho_0 (t + t_0)^{-\alpha}$, $\alpha = (1 + \gamma)h$.

3 The General case

We start by defining the action [21]

$$S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} [F(\phi)R - Z(\phi)\phi_{;\alpha}\phi^{;\alpha} - 2U(\phi)] + \mathcal{L}_M \right\}, \quad (40)$$

and therefore the FE read

$$FG_{\mu\nu} = 8\pi T_{\mu\nu}^{Matter} + Z \left[\phi_{;\mu}\phi_{;\nu} - \frac{1}{2}g_{\mu\nu}\phi_{;\alpha}\phi^{;\alpha} \right] + [F_{;\mu}F_{;\nu} - g_{\mu\nu}\square F] - U(\phi)g_{\mu\nu}, \quad (41)$$

and

$$2Z\square\phi = F_\phi R - Z_\phi\phi'^2 - 2U_\phi, \quad (42)$$

where R is the scalar curvature.

The effective gravitational constant G_{eff} between two test masses measured in laboratory Cavendish-type experiments is given by

$$G_{\text{eff}} = \frac{G_*}{F} \left[\frac{2Z(\phi)F + 4F_\phi^2}{2Z(\phi)F + 3F_\phi^2} \right]. \quad (43)$$

3.1 Matter collenations

We may define

$$T_{ij}^{\text{eff}} = \frac{1}{F(\phi)}T_{ij} + \frac{Z(\phi)}{F(\phi)} \left(\phi_{;i}\phi_{;j} - \frac{1}{2}g_{ij}\phi_{;l}\phi^{;l} \right) + \frac{1}{F(\phi)} (F_{;i}F_{;j} - g_{ij}\square F) - \frac{U(\phi)}{F(\phi)}g_{ij}, \quad (44)$$

and therefore, as above:

$$1. \quad T_1 = F(\phi)^{-1}T_{ij}$$

$$\mathcal{L}_{HO} \left(F(\phi)^{-1}T_{ij} \right) = 0 \quad \Longleftrightarrow \quad t\rho F_\phi\phi' - t\rho'F - 2\rho F = 0, \quad (45)$$

so

$$\frac{F_\phi}{F}\phi' - \frac{\rho'}{\rho} = \frac{-2}{t} \quad \Longleftrightarrow \quad \frac{\rho}{F} = t^{-2}. \quad (46)$$

$$2. \quad T_2 = \frac{Z(\phi)}{F(\phi)} \left(\phi_{;i}\phi_{;j} - \frac{1}{2}g_{ij}\phi_{;l}\phi^{;l} \right)$$

$$\mathcal{L}_{HO} \left(\frac{Z(\phi)}{F(\phi)} \left(\phi_{;i}\phi_{;j} - \frac{1}{2}g_{ij}\phi_{;l}\phi^{;l} \right) \right) = 0 \quad \Longleftrightarrow \quad t\rho F Z_\phi\phi'^2 - tZ F_\phi\phi'^2 + 2tZF\phi'' + 2ZF\phi' = 0, \quad (47)$$

thus

$$\frac{Z_\phi}{Z}\phi' - \frac{F_\phi}{F}\phi' + 2\frac{\phi''}{\phi'} = \frac{-2}{t} \quad \Longleftrightarrow \quad \frac{Z}{F}\phi'^2 = t^{-2}. \quad (48)$$

3. $T_3 = \frac{1}{F(\phi)} (F_{;\mu} F_{;\nu} - g_{\mu\nu} \square F)$ depends on the metric so we only obtain restriction on the scale factors.

4. $T_4 = \frac{U(\phi)}{F(\phi)} g_{\mu\nu}$

$$\mathcal{L}_{HO} \left(\frac{U(\phi)}{F(\phi)} g_{\mu\nu} \right) = 0 \quad \Longleftrightarrow \quad t\phi' U_\phi F - tU F_\phi \phi' + 2\phi U = 0, \quad (49)$$

and therefore

$$\frac{U_\phi}{U} \phi' - \frac{F_\phi}{F} \phi' = \frac{-2}{t} \quad \Longleftrightarrow \quad \frac{U}{F(\phi)} = t^{-2}. \quad (50)$$

3.2 Lie groups

For this model we have to solve the following equation

$$2Z\square\phi = F_\phi R - Z_\phi \phi'^2 - 2U_\phi, \quad (51)$$

that we may rewrite in the following form

$$2Z(\phi'' + ht^{-1}\phi') = Ct^{-2}F_\phi - Z_\phi \phi'^2 - 2U_\phi, \quad (52)$$

where we have assumed $\phi = \phi(t)$, and the derivatives respect t are denoted by a comma. Note that $R \approx Ct^{-2}$, with $C \in \mathbb{R}$.

The standard procedure brings us to outline the following system of PDE:

$$Z_\phi \xi_\phi - 2Z\xi_{\phi\phi} = 0, \quad (53)$$

$$(ZZ_{\phi\phi} - Z_\phi^2)\eta + ZZ_\phi\eta_\phi + 4Z^2ht^{-1}\xi_\phi + 2Z^2\eta_{\phi\phi} - 4Z^2\xi_{t\phi} = 0, \quad (54)$$

$$3(2V_\phi - Ct^{-2}F_\phi)\xi_\phi + 2ht^{-2}Z(t\xi_t - \xi) + 4Z\eta_{t\phi} - 2Z\xi_{tt} + 2Z_\phi\eta_t = 0, \quad (55)$$

$$\begin{aligned} & [Ct^{-2}(Z_\phi F_\phi - ZF_{\phi\phi}) + 2(ZU_{\phi\phi} - Z_\phi U_\phi)]\eta + Z(Ct^{-2}F_\phi - 2U_\phi)\eta_\phi + \\ & 2Z(2U_\phi - Ct^{-2}F_\phi)\xi_t + 2Ct^{-3}ZF_\phi\xi + 2Z^2(ht^{-1}\eta_t + \eta_{tt}) = 0. \end{aligned} \quad (56)$$

The symmetry $\xi = t, \eta = n\phi$, brings us to obtain the following restrictions. From Eq. (54)

$$Z_{\phi\phi} = \frac{Z_\phi^2}{Z} - Z_\phi \frac{\eta_\phi}{\eta}, \quad \implies \quad Z_{\phi\phi} = \frac{Z_\phi^2}{Z} - \frac{Z_\phi}{\phi}, \quad \implies \quad Z(\phi) = Z_0\phi^{-m}, \quad (57)$$

where $m \in \mathbb{R}$. From Eq. (56) we get

$$2(ZU_{\phi\phi} - Z_\phi U_\phi)\eta - 2ZU_\phi\eta_\phi + 4ZU_\phi\xi_t = 0, \quad (58)$$

$$Ct^{-2}[(Z_\phi F_\phi - ZF_{\phi\phi})\eta + ZF_\phi\eta_\phi - 2ZF_\phi\xi_t + 2t^{-1}ZF_\phi\xi] = 0, \quad (59)$$

and therefore

$$U_{\phi\phi} = U_\phi \frac{Z_\phi}{Z} + \left(\frac{n-2}{n} \right) \frac{U_\phi}{\phi}, \quad U_{\phi\phi} = \left(\frac{n-2}{n} - m \right) \frac{U_\phi}{\phi}, \quad (60)$$

so

$$U(\phi) = C_2 + U_0\phi^{-\frac{1}{n}(mn-2n+2)}, \quad (61)$$

while

$$F_{\phi\phi} = F_\phi \frac{Z_\phi}{Z} + \frac{F_\phi}{\phi}, \quad F_{\phi\phi} = (1-m) \frac{F_\phi}{\phi}, \quad (62)$$

obtaining

$$F(\phi) = C_1 + F_0\phi^{2-m}. \quad (63)$$

Theorem 2 *The scaling and in particular the self-similar solution admitted for the FE (41-42) have the following form*

$$\phi = \phi_0 (t + t_0)^n, \quad Z(\phi) = Z_0 \phi^{-m} = Z_0 (t + t_0)^{-nm}, \quad (64)$$

with

$$\rho = \rho_0 (t + t_0)^{-\alpha}, \quad \alpha = (1 + \gamma)h, \quad (65)$$

while

$$F(\phi) = C_1 + F_0 \phi^{2-m}, \quad U(\phi) = C_2 + U_0 \phi^{-\frac{1}{n}(mn-2n+2)} \quad (66)$$

This theorem state that we have a set of theories which admit scaling and in particular self-similar solutions if the involved functions take this particular form. The action reads

$$S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} [\phi^{2-m} R - \phi^{-m} \phi_{,\alpha} \phi^{,\alpha} - 2U(\phi)] + \mathcal{L}_M \right\}. \quad (67)$$

So setting different values for the constant m we obtain different theories. For example if $m = 0$ (induced gravity case) then the action (40) yields

$$S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} [\phi^2 R - \phi_{,\alpha} \phi^{,\alpha} - 2U(\phi)] + \mathcal{L}_M \right\},$$

$$\phi = \phi_0 (t + t_0)^n, \quad Z(\phi) = Z_0, \quad F(\phi) = F_0 \phi^2, \quad U(\phi) = U_0 \phi^{\frac{1}{n}(2n-2)}, \quad G_{\text{eff}} \approx \phi^{-2},$$

and if $m = 1$ (usual JBD theory with a potential) then the action (40) yields

$$S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left[\phi R - \frac{\omega}{\phi} \phi_{,\alpha} \phi^{,\alpha} - 2U(\phi) \right] + \mathcal{L}_M \right\}.$$

$$\phi = \phi_0 (t + t_0)^n, \quad Z(\phi) = Z_0 \phi^{-1}, \quad F(\phi) = F_0 \phi, \quad U(\phi) = U_0 \phi^{\frac{1}{n}(n-2)}, \quad G_{\text{eff}} \approx \phi^{-1},$$

i.e.

$$Z(\phi) = \frac{\omega(\phi)}{\phi}, \quad \omega(\phi) = \text{const.}$$

and to end if $m = 2$, then the action (40) yields

$$S = \frac{1}{8\pi} \int d^4x \sqrt{-g} \left\{ \frac{1}{2} \left[R - \frac{\omega}{\phi^2} \phi_{,\alpha} \phi^{,\alpha} - 2U(\phi) \right] + \mathcal{L}_M \right\}.$$

we get:

$$\phi = \phi_0 (t + t_0)^n, \quad Z(\phi) = Z_0 \phi^{-2}, \quad F(\phi) = F_0, \quad U(\phi) = U_0 \phi^{-\frac{2}{n}}, \quad G_{\text{eff}} \approx \text{const..}$$

This particular case is very similar to the scalar field cosmological model with $G_{\text{eff}} \approx \text{const.}$ Notice that this model is quite different from the Barker's theory [25].

4 Chameleon cosmology

We begin with the BD chameleon theory in which the scalar field is coupled non-minimally to the matter field via the action

$$S = \int d^4x \sqrt{-g} \left(\phi R - \frac{\omega}{\phi} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - U(\phi) + J(\phi) L_m \right), \quad (68)$$

where R is the Ricci scalar curvature, ϕ is the BD scalar field with a potential $U(\phi)$. The chameleon field ϕ is non-minimally coupled to gravity, ω is the dimensionless BD parameter. The last term in the action indicates the interaction between the matter Lagrangian L_m and some arbitrary function $J(\phi)$ of the BD scalar field. In the limiting case $J(\phi) = 1$, we obtain the standard BD theory.

The gravitational field equations derived from the action (68) with respect to the metric is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{J(\phi)}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2} \left(\phi_\mu \phi_\nu - \frac{1}{2}g_{\mu\nu}\phi^\alpha \phi_\alpha \right) + \frac{1}{\phi}[\phi_{\mu;\nu} - g_{\mu\nu}\square\phi] - g_{\mu\nu}\frac{U(\phi)}{2\phi}. \quad (69)$$

The Klein-Gordon equation (or the wave equation) for the scalar field is

$$(2\omega + 3)\square\phi = T \left(J - \frac{1}{2}\phi J_{,\phi} \right) + (\phi U_{,\phi} - 2U). \quad (70)$$

Similarly the energy conservation for the cosmic fluid is

$$\dot{\rho} + \theta(\rho + p) = 0, \quad \theta = u^i_{;i}. \quad (71)$$

We shall use the equation of state (EoS) for the fluid $p = \gamma\rho$, thus (71) yields $\rho = \rho_0(t + t_0)^{-\alpha}$.

4.1 Matter collineations

By defining

$$T_{ij}^{\text{eff}} = \frac{J(\phi)}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2} \left(\phi_\mu \phi_\nu - \frac{1}{2}g_{\mu\nu}\phi^\alpha \phi_\alpha \right) + \frac{1}{\phi}[\phi_{\mu;\nu} - g_{\mu\nu}\square\phi] - g_{\mu\nu}\frac{U(\phi)}{2\phi}, \quad (72)$$

as it is observed it is only necessary to calculate the first component, the rest of them have been already calculated in the above sections.

If $T_1 = F(\phi)\phi^{-1}T_{ij}$, then

$$\mathcal{L}_{HO} \left(\frac{J(\phi)}{\phi}T_{ij} \right) = 0, \quad (73)$$

yields

$$tJ_\phi\phi'p\phi - t\rho J\phi' + t\rho'J\phi + 2\rho J\phi = 0, \quad (74)$$

algebra brings us to get:

$$\frac{J_\phi\phi'}{J} + \frac{\rho'}{\rho} - \frac{\phi'}{\phi} = -\frac{2}{t} \quad \Longleftrightarrow \quad \frac{J}{\phi}\rho = t^{-2}. \quad (75)$$

where $J' = J_\phi\phi'$, but we do not obtain more information about the behaviour of some of the functions ϕ or $J(\phi)$.

4.2 Lie Groups

We need to study the following ODE

$$\phi'' + ht^{-1}\phi' = C \left(J - \frac{1}{2}\phi J_\phi \right) t^{-\alpha} + K(2V - \phi U_\phi), \quad (76)$$

where, $C = \frac{8\pi(1-3\gamma)}{3+2\omega}$, $K = \frac{1}{3+2\omega}$, $h = \text{const.}$,

Therefore the standard procedure brings us to outline the following system of PDE

$$\xi_{\phi\phi} = 0, \quad (77)$$

$$ht^{-1}\xi_\phi + \eta_{\phi\phi} - 2\xi_{\phi t} = 0, \quad (78)$$

$$ht^{-2}(t\xi_t - \xi) + 2\eta_{t\phi} - \xi_{tt} - 3 \left[Ct^{-\alpha} \left(J - \frac{1}{2}\phi J_\phi \right) + 2U - \phi U_\phi \right] \xi_\phi = 0, \quad (79)$$

$$\begin{aligned} & \eta_{tt} + ht^{-1}\eta_t + 2\left(\phi U_\phi - U - Ct^{-\alpha}\left(J - \frac{1}{2}\phi J_\phi\right)\right)\xi_t + \alpha Ct^{-\alpha-1}\left(J - \frac{1}{2}\phi J_\phi\right)\xi \\ & + \left(2V - \phi V_\phi + Ct^{-\alpha}\left(J - \frac{1}{2}\phi J_\phi\right)\right)\eta_\phi - \left(\frac{Ct^{-\alpha}}{2}(J_\phi - \phi J_{\phi\phi}) + U_\phi - \phi U_{\phi\phi}\right)\eta = 0, \end{aligned} \quad (80)$$

As we already know, from the MC approach, it must be verified the relationship

$$\frac{J(\phi)}{\phi}\rho = t^{-2}, \quad \implies \quad \frac{J(\phi)}{\phi} = t^{-2+\alpha}. \quad (81)$$

algebra brings us to get

$$Ct^{-\alpha}\left[\left(J - \frac{1}{2}\phi J_\phi\right)(\alpha t^{-1}\xi - 2\xi_t + \eta_\phi) - \frac{1}{2}(J_\phi - \phi J_{\phi\phi})\eta\right] = 0, \quad (82)$$

$$(U_\phi - \phi U_{\phi\phi})\eta + (2U - \phi U_\phi)(-2\xi_t + \eta_\phi) = 0. \quad (83)$$

For example the symmetry $\xi = t, \eta = n\phi$, brings us to obtain the following restriction on $U(\phi)$. From Eq. (83) we obtain the ODE

$$U_{\phi\phi} = 2\left(1 - \frac{1}{n}\right)\frac{U_\phi}{\phi} + 2\left(\frac{2}{n} - 1\right)\frac{U}{\phi^2}, \quad (84)$$

whose solution has been obtained in section 3, i.e. $U(\phi) = U_0\phi^{\frac{1}{n}(n-2)}$. From Eq. (82) we get the next ODE for J :

$$J_{\phi\phi} = \left(\frac{\alpha + 2n - 2}{n}\right)\frac{J_\phi}{\phi} - 2\left(\frac{\alpha + n - 2}{n}\right)\frac{J}{\phi^2}, \quad (85)$$

finding that the most general solution is

$$J(\phi) = J_0\phi^{\frac{1}{n}(n+\alpha-2)} + C_2\phi^2. \quad (86)$$

Theorem 3 *The scaling and in particular the self-similar solution admitted for the FE (41-42) have the following form*

$$\phi = \phi_0(t+t_0)^n \quad \implies \quad U(\phi) = U_0\phi^{\frac{1}{n}(n-2)} = U_0(t+t_0)^{n-2}, \quad (87)$$

with $\rho = \rho_0(t+t_0)^{-\alpha}$, $\alpha = (1+\gamma)h$, while

$$J(\phi) = J_0\phi^{\frac{1}{n}(n+\alpha-2)} = J_0(t+t_0)^{n+\alpha-2}. \quad (88)$$

5 The Kantowski-Sach model

We start by considering the Killing vector fields (KVF)

$$Z_1 = \partial_y, \quad Z_2 = \cot z \cos y \partial_y + \sin y \partial_z, \quad Z_3 = -\cot z \sin y \partial_y + \cos y \partial_z, \quad Z_4 = \partial_x, \quad (89)$$

such that

$$[Z_1, Z_2] = Z_3, \quad [Z_2, Z_3] = Z_1, \quad [Z_3, Z_1] = Z_2, \quad [Z_4, Z_i] = 0, \quad (90)$$

in such a way that the metric takes the following form

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)(\sin^2 z dy^2 + dz^2), \quad (91)$$

We find that the metric (91) admits the following HVF

$$H = (t+t_0)\partial_t + \left(1 - (t+t_0)\frac{a'}{a}\right)x\partial_x + \left(1 - (t+t_0)\frac{b'}{b}\right)y\partial_y + \left(1 - (t+t_0)\frac{b'}{b}\right)z\partial_z, \quad (92)$$

where the scale factors behave as

$$b(t) = b_0(t + t_0) \quad \text{and} \quad a(t) = a_0(t + t_0)^m, \quad \forall m \in \mathbb{R}^+. \quad (93)$$

Taking into account these results we shall calculate some exact cosmological solutions. It is a straightforward task to carry out the calculations for this reason we only show the results. The detailed exposition of the followed method may be found for example in [27].

5.1 Vacuum solution

There is no SS vacuum solution for this model

5.2 Perfect fluid model

The stress-energy tensor is defined by, $T_{ij} = (\rho + p)u_i u_j - p g_{ij}$, and where we are taking into account the conservation principle so, $T_{ij}^{;j} = 0$, and the equation of state $p = \gamma\rho$, ($\gamma = \text{const.}$). We find that

$$a(t) = a_0(t + t_0)^{\sqrt{2}}, \quad b(t) = b_0(t + t_0), \quad \rho = \rho_0(t + t_0)^{-2}, \quad (94)$$

and this solution is only valid for

$$\gamma_c = 1 - \sqrt{2} \approx -0.41421356. \quad (95)$$

Note that our solution accelerates since

$$q = \frac{3}{h} - 1 = \frac{3}{2 + \sqrt{2}} - 1 < 0. \quad (96)$$

5.3 Time-varying constant model

In this model we consider the constants G and Λ as time varying functions in such a way that the FE are:

$$G_{ij} = G(t)T_{ij} - \Lambda(t)g_{ij}, \quad (G(t)T_{ij} - \Lambda(t)g_{ij})^{;j} = 0, \quad (97)$$

with the constrain $T_{ij}^{;j} = 0$.

We have found the next results

$$\begin{aligned} a(t) &= a_0(t + t_0)^{\sqrt{2}}, & b(t) &= b_0(t + t_0), & \rho &= \rho_0(t + t_0)^{-(\gamma+1)(2+\sqrt{2})} \\ G &= G_0(t + t_0)^{(\gamma+1)(2+\sqrt{2})-2}, & \Lambda &= \Lambda_0(t + t_0)^{-2}, \end{aligned} \quad (98)$$

where

$$G \approx \begin{cases} \text{decreasing } \forall \gamma \in [-1, \gamma_c) \\ \text{constant if } \gamma = \gamma_c \\ \text{increasing } \forall \gamma \in (\gamma_c, 1] \end{cases} \quad \Lambda_0 \approx \begin{cases} \text{negative } \forall \gamma \in [-1, \gamma_c) \\ \text{vanish if } \gamma = \gamma_c \\ \text{positive } \forall \gamma \in (\gamma_c, 1] \end{cases} \quad (99)$$

where γ_c is given by Eq. (95). This solution is valid for all EoS γ , i.e. there are no restrictions on the γ parameter. As above $q < 0$.

5.4 JBD model with CC

For this model (this corresponds to the exposed one in section 2) the FE are as follows

$$R_{ij} - \frac{1}{2}g_{ij}R = \frac{8\pi}{c^4\phi}T_{ij} + \Lambda(\phi)g_{ij} + \frac{\omega}{\phi^2}\left(\phi_{,i}\phi_{,j} - \frac{1}{2}g_{ij}\phi_{,l}\phi^{,l}\right) + \frac{1}{\phi}\left(\phi_{;ij} - g_{ij}\square\phi\right), \quad (100)$$

$$(3 + 2\omega(\phi))\square\phi = 8\pi T - \omega_\phi\phi_{,l}\phi^{,l} - 2\phi(\phi\Lambda_\phi - \Lambda(\phi)), \quad (101)$$

where $T = T_i^i$ is the trace of the stress-energy tensor. The gravitational coupling $G_{\text{eff}}(t)$ is given by

$$G_{\text{eff}}(t) = \left(\frac{2\omega+4}{2\omega+3}\right) \frac{G_*}{\phi(t)}. \quad (102)$$

From the stated theorem of section 2 we already know that the physical quantities behave as follows:

$$\begin{aligned} \phi &= \phi_0(t+t_0)^n, & \implies & \Lambda(\phi) = \Lambda_0\phi^{\frac{-1}{n}(n+\alpha)} = \Lambda_0(t+t_0)^{-(n+\alpha)}, & \text{with } n+\alpha &= 2. \\ \omega(\phi) &= \text{const.}, & \text{and } \rho &= \rho_0(t+t_0)^{-\alpha}, & \alpha &= (1+\gamma)h, & h &= 2+m. \end{aligned}$$

We have found the next solution. The scale factors behave as

$$a(t) = a_0(t+t_0)^m, \quad b(t) = b_0(t+t_0), \quad (103)$$

with

$$m = \frac{1}{2\gamma}(1-\gamma-A) \in [1.2361, 4], \quad \forall \gamma \in \left[-1, \frac{1}{9}\right], \quad m_{\gamma=1} = 0, \quad (104)$$

with $A = \sqrt{9\gamma^2 - 10\gamma + 1}$. m is not defined $\forall \gamma \in (\frac{1}{9}, 1)$.

$$\phi = \phi_0(t+t_0)^n, \quad G_{\text{eff}} \approx \phi^{-1}, \quad (105)$$

where

$$\begin{aligned} n &= \frac{1}{2\gamma}(-1-3\gamma^2+A(1+\gamma)) \in \left[-\frac{14}{3}, 2\right], & \forall \gamma &\in \left[-1, \frac{1}{9}\right], \\ n &> 0, \forall \gamma \in [-1, \gamma_c), & n_{\gamma_c} &= 0, & n < 0, \forall \gamma \in \left(\gamma_c, \frac{1}{9}\right], & n_{\gamma=1} &= -2, \end{aligned} \quad (106)$$

note that γ_c is given by Eq. (95).

$$\begin{aligned} \phi_0 &= -\frac{1+4\gamma^2-7\gamma+A(2\gamma-1)}{\gamma(-1+3\gamma+A)} \in \left[-3, \frac{11}{3}\right], & \forall \gamma &\in \left[-1, \frac{1}{9}\right], \\ \phi_0 &< 0 \forall \gamma \in [-1, \gamma_1), & \phi_{0\gamma_1} &= 0, & \phi_0 > 0, \forall \gamma \in \left(\gamma_1, \frac{1}{9}\right], & \phi_{0\gamma=1} &= 1, \end{aligned} \quad (107)$$

where $\gamma_1 = -0.1708203932$. Therefore this solution is only valid if $\gamma \in (\gamma_1, \frac{1}{9}]$ and $\gamma = 1$.

$$\Lambda(\phi) = \Lambda_0(t+t_0)^{-2}, \quad \Lambda_0 = \Lambda_0(\gamma, \omega), \quad \omega(\phi) = \text{const} = 10^4,$$

(the recent value of $\omega(\phi)$ has been obtained from [28]) where the performed numerical analysis shows us that: $\Lambda_0 > 0 \forall \gamma \in (-1, \gamma_2)$, $\Lambda_{0\gamma_2} = 0$, $\Lambda_0 > 0 \forall \gamma \in (\gamma_2, \gamma_3)$, $\Lambda_{0\gamma_c} = 0$, $\Lambda_0 > 0 \forall \gamma \in (\gamma_3, \frac{1}{9}]$, and $\Lambda_{0\gamma=1} = 1$, where $\gamma_2 = -0.4142736105$, and $\gamma_c = -0.4142135624$. Note that $\gamma_1 \in (\gamma_3, \frac{1}{9}]$, and then $\Lambda_{0\gamma_1} > 0$.

$$\rho = \rho_0(t+t_0)^{-\alpha}, \quad \alpha = (1+\gamma)(m+2), \quad \rho_0 = \rho_0(\gamma, \omega), \quad (108)$$

$$\begin{aligned} \rho_0 > 0 \forall \gamma \in (-1, \gamma_4), \quad \rho_{0\gamma_4} = 0, \quad \rho_0 < 0 \forall \gamma \in (\gamma_4, \gamma_5), \quad \rho_{0\gamma_5} = 0, \\ \rho_0 > 0 \forall \gamma \in (\gamma_5, \gamma_1), \quad \rho_{0\gamma_1} = 0, \quad \rho_0 < 0, \forall \gamma \in \left(\gamma_1, \frac{1}{9}\right], \quad \rho_{0\gamma=1} = -795.8940815, \end{aligned} \quad (109)$$

where $\gamma_4 = -0.4187505363$, $\gamma_5 = -0.4097441700$, and $\gamma_1 = -0.1708203932$.

Therefore this solution is unphysical, since $\phi_0 > 0$, $\forall \gamma \in (\gamma_1, \frac{1}{9}]$ but $\rho_0 < 0$, $\forall \gamma \in (\gamma_1, \frac{1}{9}]$. Notice that $m_{\gamma_c} = \sqrt{2}$, $n_{\gamma_c} = 0$, and $\Lambda_{0\gamma_c} = 0$ as in the above solution.

5.5 JBD model with potential

For this model (which corresponds to the particular model, $m = 1$, exposed in section 3) the FE are as follows

$$G_{ij} = \frac{8\pi}{c^4 \phi} T_{ij} + \frac{\omega}{\phi^2} \left(\phi_{,i} \phi_{,j} - \frac{1}{2} g_{ij} \phi_{,l} \phi^{,l} \right) + \frac{1}{\phi} \left(\phi_{,ij} - g_{ij} \square \phi \right) + \frac{U(\phi)}{\phi} g_{ij}, \quad (110)$$

$$(3 + 2\omega(\phi)) \square \phi = 8\pi T - \omega_\phi \phi_{,l} \phi^{,l} + \phi U_\phi - 2U(\phi), \quad (111)$$

where $T = T_i^i$ is the trace of the stress-energy tensor. The gravitational coupling $G_{\text{eff}}(t)$ is given by

$$G_{\text{eff}}(t) = \left(\frac{2\omega + 4}{2\omega + 3} \right) \frac{G_*}{\phi(t)}. \quad (112)$$

$$\begin{aligned} \phi = \phi_0 (t + t_0)^n \quad \implies \quad U(\phi) = U_0 \phi^{\frac{1}{n}(n-2)} = U_0 (t + t_0)^{(n-2)}, \quad \text{with} \quad n + \alpha = 2. \quad G_{\text{eff}} \approx \phi^{-1}, \\ \omega(\phi) = \text{const.} \quad \text{and} \quad \rho = \rho_0 (t + t_0)^{-\alpha}, \quad \alpha = (1 + \gamma)h, \quad h = 2 + m \end{aligned}$$

We have found the following solution. The scale factors behave as

$$a(t) = a_0 (t + t_0)^m, \quad b(t) = b_0 (t + t_0), \quad (113)$$

with

$$m = \frac{1}{2\gamma} (1 - \gamma - A) \in [1.2361, 4], \quad \forall \gamma \in \left[-1, \frac{1}{9}\right], \quad m_{\gamma=1} = 0 \quad (114)$$

with $A = \sqrt{9\gamma^2 - 10\gamma + 1}$. m is not defined $\forall \gamma \in (\frac{1}{9}, 1)$.

$$\phi = \phi_0 (t + t_0)^n, \quad G_{\text{eff}} \approx \phi^{-1} = G_* (t + t_0)^{-n}, \quad (115)$$

where

$$\begin{aligned} n = \frac{1}{2\gamma} (-1 - 3\gamma^2 + A(1 + \gamma)) \in \left[-\frac{14}{3}, 2\right], \quad \forall \gamma \in \left[-1, \frac{1}{9}\right], \\ n > 0, \forall \gamma \in [-1, \gamma_c), \quad n_{\gamma_c} = 0, \quad n < 0, \forall \gamma \in \left(\gamma_c, \frac{1}{9}\right], \quad n_{\gamma=1} = -2, \end{aligned} \quad (116)$$

note that γ_c is given by Eq. (95), while

$$\phi_0 = \frac{4\gamma(3\gamma - 1 + A)}{5\gamma^2 + 4\gamma - 1 + A(\gamma + 1)}, \quad \phi_0 > 0, \forall \gamma \in \left[-1, \frac{1}{9}\right], \quad \phi_{0\gamma=1} = 1, \quad (117)$$

Therefore this solution is only valid if $\gamma \in [-1, \frac{1}{9}]$ and $\gamma = 1$.

$$U(\phi) = U_0 (t + t_0)^{n-2}, \quad U_0 = U_0(\gamma, \omega), \quad \omega(\phi) = \text{const} = 10^4, \quad (118)$$

$$U_0 > 0 \forall \gamma \in \left(-1, \frac{1}{9}\right] \setminus (\gamma_2, \gamma_c), \quad U_{0\gamma=1} = 1,$$

$$U_{0\gamma_2} = 0, \quad U_0 < 0 \forall \gamma \in (\gamma_2, \gamma_c), \quad U_{0\gamma_3} = 0,$$

where $\gamma_2 = -0.4142736105$, and $\gamma_c = -0.4142135624$. Note that $(n-2) < 0$.

$$\rho = \rho_0 (t+t_0)^{-\alpha}, \quad \alpha = (1+\gamma)(m+2), \quad \rho_0 = \rho_0(\gamma, \omega), \quad (119)$$

$$\rho_0 < 0 \forall \gamma \in \left(-1, \frac{1}{9}\right] \setminus (\gamma_4, \gamma_5), \quad \rho_{0\gamma=1} = -795.8940815,$$

$$\rho_{0\gamma_4} = 0, \quad \rho_0 > 0 \forall \gamma \in (\gamma_4, \gamma_5), \quad \rho_{0\gamma_5} = 0,$$

where $\gamma_4 = -0.4187505363$, $\gamma_5 = -0.4097441700$.

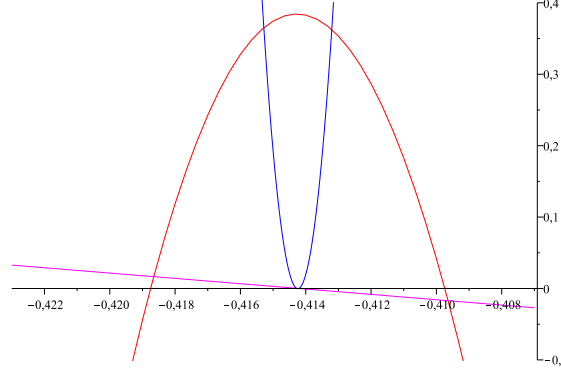


Figure 1: JBD model with potential. Solution $\forall \gamma \in (\gamma_4, \gamma_5)$. ρ_0 is plotted in red color. U_0 in blue and n in magenta color.

Therefore this solution is only valid $\forall \gamma \in (\gamma_4, \gamma_5) \ni \gamma_c$, where $\rho_0 > 0$ and $\phi_0 > 0$ while $U_0 > 0 \forall \gamma \in (\gamma_4, \gamma_5) \setminus (\gamma_2, \gamma_c)$, see fig. (1). Notice that $G_{\text{eff}}(\gamma)$ is decreasing if $\gamma \in (\gamma_4, \gamma_c)$, constant if $\gamma = \gamma_c$ and growing if $\gamma \in (\gamma_c, \gamma_5)$. As in the above solutions we have that that $m_{\gamma_c} = \sqrt{2}$, $n_{\gamma_c} = 0$, and $\Lambda_{0\gamma_c} = 0$. We also emphasize that $q < 0$, $\forall \gamma \in (\gamma_4, \gamma_5)$. To end we have calculated the values of m in γ_4 and γ_5 , they are: $m_{\gamma_4} = 1.4117477876367817948$ and $m_{\gamma_5} = 1.4166732719815729598$. These values will acquire a complete sense in the next model.

5.6 Chameleon JBD model

This model corresponds to the exposed one in section 4. In this case FE read:

$$G_{ij} = \frac{J(\phi)}{\phi} T_{ij} + \frac{\omega}{\phi^2} \left(\phi_{,i} \phi_{,j} - \frac{1}{2} g_{ij} \phi_{,l} \phi^{,l} \right) + \frac{1}{\phi} \left(\phi_{;ij} - g_{ij} \square \phi \right) + \frac{U(\phi)}{\phi} g_{ij}, \quad (120)$$

$$(2\omega + 3) \square \phi = T \left(J - \frac{1}{2} \phi J_\phi \right) - (\phi U_\phi - 2U), \quad (121)$$

where the main quantities behave as follows

$$\phi = \phi_0 (t+t_0)^n, \quad U(\phi) = U_0 \phi^{\frac{1}{n}(n-2)} = U_0 (t+t_0)^{(n-2)}, \quad G_{\text{eff}} \approx \phi^{-1},$$

$$\omega(\phi) = \text{const.} \quad \text{and} \quad \rho = \rho_0 (t+t_0)^{-\alpha}, \quad \alpha = (1+\gamma)h, \quad h = 2+m, \quad (122)$$

$$J(\phi) = J_0 \phi^{\frac{\alpha+n-2}{n}} = J_0 (t+t_0)^{\alpha+n-2}.$$

We have obtained the following solution. $n = n(m)$;

$$n = -\frac{m^2 - 2}{m - 1}, \quad (123)$$

in such a way that $n = 0$, iff $m_c = \pm 1.414213562 = \pm\sqrt{2}$, we only consider the positive solution, so, $m \in \mathbb{R}^+ \setminus \{1\}$. $n < 0$ if $m \in [0, 1) \cup (m_{c+}, \infty)$, and $n > 0$ if $m \in (1, m_{c+})$.

The rest of the quantities depend on (m, γ, ω) , so in order to carry out the numerical analysis it is necessary to fix the value of ω and γ . Setting $\omega = 4 \cdot 10^4$, we have studied some equation of state, $\gamma = 1, 1/3, 0, -1/3$, i.e. the usual ones.

For $\gamma = 1$ then; $\phi = \phi_0(t + t_0)^n$, and then, $G_{\text{eff}} \approx \phi^{-1} = G_*(t + t_0)^{-n}$, where

$$\phi_0 = \phi_0(m, \gamma = 1, \omega \approx 10^4) = -\frac{(10002m^4 - 2m^3 - 60007m^2 + 80010)}{(5001m^4 - m^3 - 9999m^2 - 3m + 2)}, \quad (124)$$

then

$$\phi_0 = 0, \quad \Longleftrightarrow \quad m_1 = 1.414184279, \quad m_2 = 2.000024995, \quad (125)$$

and ϕ_0 is not defined when

$$m_{a_1} = 0.1399429180, \quad m_{a_2} = 1.414180724,$$

in such a way that $\phi_0 > 0$ if $m \in (m_{a_1}, m_{a_2})$. $\phi_0 < 0$ if $m \in (m_{a_2}, m_1)$, and $\phi_0 > 0$ if $m \in (m_1, m_2)$. U_0 behaves as

$$U_0 = \phi_0 \frac{(m^2 - 2m + 2)}{2(m - 1)^2}, \quad (126)$$

therefore U_0 has the same roots than ϕ_0 and it is not defined when

$$m_{a_1} = 0.1399429180, \quad m_{a_3} = 1, \quad m_{a_2} = 1.414180724,$$

in such a way that $U_0 > 0$ if $m \in (m_{a_1}, m_{a_2}) \setminus \{m_{a_3}\}$. $U_0 < 0$ if $m \in (m_{a_2}, m_1)$, and $U_0 > 0$ if $m \in (m_1, m_2)$.

$\rho_0 = \rho_0(m, \gamma = 1/3, \omega \approx 10^4, J_0 = 10)$ behaves as

$$\rho_0 = \phi_0 \frac{(10002m^4 - 2m^3 - 40003m^2 - 2m + 40006)}{160\pi(5001m^4 - m^3 - 9999m^2 - 3m + 2)(m - 1)^2},$$

then $\rho_0 = 0$, iff

$$m_1 = 1.414184279, \quad m_2 = 2.000024995, \quad m_3 = 1.411747788, \quad m_4 = 1.416673272,$$

and it is not defined if

$$m_{a_1} = 0.1399429180, \quad m_{a_3} = 1, \quad m_{a_2} = 1.414180724,$$

finding therefore that $\rho_0 < 0$, if $m \in (m_{a_1}, m_{a_2}) \setminus \{m_{a_3}\}$, $\rho_0 > 0$, $\forall m \in (m_3, m_4)$ and $\rho_0 < 0$, if $m \in (m_4, m_2)$. Nevertheless a careful analysis shows us that ρ_0 is not defined when $m_{a_2} = 1.414180724$, note that $m_{a_2} \in (m_3, m_4)$. Thus $\rho_0 > 0$, $\forall m \in (m_3, m_{a_2}) \cup (m_1, m_4)$, if $m \in (m_{a_2}, m_1)$ then $\rho_0 < 0$.

Note that

$$m_{a_1} = 0.1399429180 < m_{a_3} = 1 < m_3 = 1.411747788 < m_{a_2} = 1.414180724 < \\ < m_1 = 1.414184279 < m_{c+} = 1.414213562 < m_4 = 1.416673272 < m_2 = 2.000024995.$$

Therefore this solution is only valid if $m \in (m_3, m_{a_2}) \cup (m_1, m_4)$, since in this interval $\rho_0 > 0, \phi_0 > 0$. In fig. (2) we have plotted this situation.

In order to clarify and to compare (with the following results) the results we have write in the following table the roots of the constants (n, ϕ_0, U_0, ρ_0) :

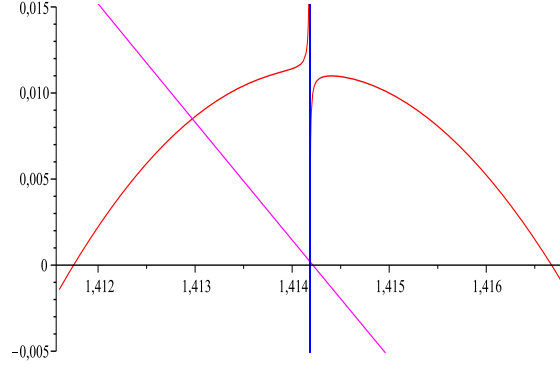


Figure 2: Chameleon JBD model with $\gamma = 1$ and $m \in (m_3, m_4)$. ρ_0 is plotted in red color. U_0 in blue and n in magenta color. ϕ_0 is plotted in green color but it appears under the graph of U_0 .

	r_1	r_2	r_3	r_4
n			$m_c = 1.414213562$	
ϕ_0		$m_1 = 1.414184279$,		
U_0		$m_1 = 1.414184279$,		
ρ_0	$m_3 = 1.411747788$	$m_1 = 1.414184279$,		$m_4 = 1.416673272$

The cases $\gamma = 1/3, 0$ and $-1/3$, are quite similar. For example if $\gamma = 1/3$, the performed numerical analysis shows us that the roots of the constants are as follows (compare with the above table)

	r_1	r_2	r_3	r_4
n			$m_c = 1.414213562$	
ϕ_0		$m_1 = 1.414197143$	$m_c = 1.414213562$	
U_0		$m_1 = 1.414197143$,	$m_c = 1.414213562$	
ρ_0	$m_3 = 1.411747788$	$m_1 = 1.414197143$,	$m_c = 1.414213562$	$m_4 = 1.416673272$

and therefore we have the following scenario, see fig. 3 where we have plotted the interval $m \in (m_3, m_4)$. Note the analogies with regard to the case $\gamma = 1$.

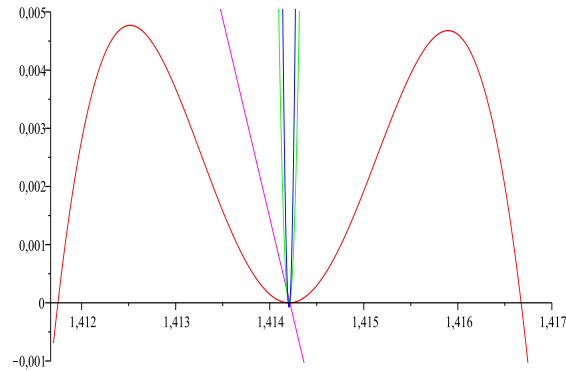


Figure 3: Chameleon JBD model with $\gamma = 1/3$ and $m \in (m_3, m_4)$. ρ_0 is plotted in red color. U_0 in blue and n in magenta color. ϕ_0 is plotted in green color.

In fig. 4 we have plotted in detail the interval $m \in (m_1, m_c)$.

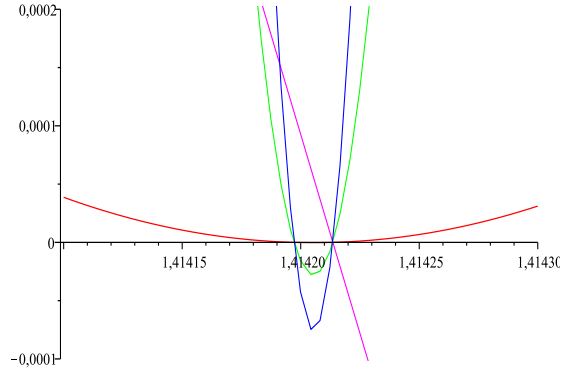


Figure 4: Chameleon JBD model with $\gamma = 1/3$ and $m \in (m_1, m_c)$. ρ_0 is plotted in red color. U_0 in blue and n in magenta color. ϕ_0 is plotted in green color.

Therefore this solution is valid for any value of γ , while in the above solution it was only valid for a small interval of γ_c . Remember the solution in the last model, when $m_{\gamma_4} = 1.4117477876367817948$ and $m_{\gamma_5} = 1.4166732719815729598$.

5.7 Induced Gravity model

In this model the FE read

$$F(\phi) G_{ij} = T_{ij} + \left(\phi_{;i} \phi_{;j} - \frac{1}{2} g_{ij} \phi_{;l} \phi^{;l} \right) + \left(F(\phi)_{;ij} - g_{ij} \square F(\phi) \right) + U(\phi) g_{ij}, \quad (127)$$

$$2\square\phi = RF_\phi - 2U_\phi, \quad (128)$$

where R is the scalar curvature and $F(\phi) = \phi^2/4\omega$. The gravitational coupling $G_{\text{eff}}(t)$ is given by

$$G_{\text{eff}}(t) \approx \frac{G_*}{\phi^2}. \quad (129)$$

For the metric (91) the scalar curvature is given by: $R = 2 \left(\frac{a''}{a} + 2 \frac{b'a'}{ba} + 2 \frac{b''}{b} + \frac{1}{b^2} + \left(\frac{b'}{b} \right)^2 \right)$. As we already know, the physical quantities behave as follows:

$$\begin{aligned} \phi &= \phi_0 (t+t_0)^n, & U(\phi) &= U_0 \phi^{\frac{2}{n}(n-1)} = U_0 (t+t_0)^{2(n-1)}, & G_{\text{eff}} &\approx \phi^{-2}, \\ \omega(\phi) &= \text{const.}, & \text{and} & & \rho &= \rho_0 (t+t_0)^{-\alpha}, & \alpha &= (1+\gamma)h, & h &= 2+m. \end{aligned}$$

We have found the following solution. The scale factors behave as

$$a(t) = a_0 (t+t_0)^m, \quad b(t) = b_0 (t+t_0),$$

with $m = m(n)$:

$$m = -n + A, \quad A = \sqrt{2n^2 + n + 2},$$

where, as it is observed if $n = 0$, then $m = \sqrt{2}$. $\phi_0 = \phi_0(n, \gamma)$ behaves as follows

$$\phi_0 = \frac{2n [3n^2(1+\gamma) + n(1-2A + \omega(1+A)(1+\gamma) - (2A+1)\gamma) + 3(\gamma+1)]}{n^3(2+\omega-\gamma(1+\omega)) + n^2(\gamma(2+\omega+A) - \omega - 1) - \gamma(A+1) - 1},$$

note that $\phi_0 = 0$, iff $\gamma = -1$, and $n \neq 0$. With regard to the constant $U_0 = U_0(n, \gamma, \omega)$ we have found the next value

$$U_0 = \frac{\phi_0}{\omega(n-1)} (n(A-n)(\omega-2) + (n^2+1)(\omega+1)+3).$$

The constant for the energy density, $\rho_0 = \rho_0(n, \gamma, \omega)$, behaves as follows:

$$\rho_0 = \frac{\phi_0}{2\omega(n-1)} (\phi_0 (n\omega - (1+n) + (n-A)(1-n^2)) - 2n^3(\omega+1) + 2n^2(-1+2(A-n)(1-\omega)-\omega) - 6n).$$

Since the solutions depend on the parameters (n, γ, ω) then we need to fix them. For example, if we set For $\omega \approx 4 \cdot 10^4$, then the solutions only depend on (n, γ) in such a way that given different values to n then we shall may to study their behaviour. For $n = -1$, $\omega \approx 4 \cdot 10^4$ we get $\phi_0 > 0$ and $\rho_0 > 0$ iff $\gamma < -1$ while $U_0 < 0$ if $\gamma < -1$. We arrive at the same conclusion if $n = -2$. Note that the cases $n = 0$ and $n = 1$ are forbidden. Therefore, the obtained solution is only valid if $\gamma < -1$.

6 Conclusions

We have studied under the self-similar hypothesis the admitted form of the different unknown functions in several scalar-tensor theories. By employing the matter collineation (MC) approach, i.e. calculating the Lie derivate of the effective stress-energy tensor with respect to an HVF, and the Lie group method, we have been able to state theorems valid for all the Bianchi geometries as well as for a flat FRW metric. We have used both tactics, because with the MC in some of the models studied we are only able to obtain relationships between the physical quantities. With this tactic we only obtain self-similar solutions. Nevertheless, with the Lie group method we are able to obtain the exact form for each of the physical quantities. Furthermore, with this approach we obtain scaling solutions which are more general than the self-similar one.

In the first of the models studied we arrive to the conclusion that $\phi \approx (t+t_0)^n$, and therefore $G_{\text{eff}} \approx (t+t_0)^{-n}$, while the dynamical cosmological constant behaves as $\Lambda \approx (t+t_0)^{-2}$. In the same way we have deduced that the Brans-Dicke parameter $\omega_{BD}(\phi)$ must be constant. In the second of the models studied, the generalized scalar-tensor model, the dynamical cosmological constant is mimicked by the potential $U(\phi)$, and therefore we have three unknown functions, $F(\phi)$, $Z(\phi)$ and the potential $U(\phi)$. We arrive at a very general result which allows us to outline different scalar-tensor models that admit self-similar solutions. Actually this result is in agreement with the fact that we may pass from one model to another through conformal transformations. As an example we have emphasized three relevant models, the standard scalar-tensor model, the induced gravity model and a specific model which is very similar to the scalar cosmological model where $G_{\text{eff}} \approx \text{const}$. In the third of the models studied, following the same procedure, we calculate the admitted form for the unknown functions $J(\phi)$ and the potential in order to obtain scaling and self-similar solutions.

Once we have established all these results then we study a particular example, the Kantowski-Sach model. The same procedure may be applied to other Bianchi models. We begin by showing that this metric admits an HVF. We explore some cosmological models and applying the stated theorems we find exact solutions. We show that there is no vacuum solution. For the perfect fluid case, within the general relativity framework, we find that the solution obtained is only valid for a particular equation of state, γ_c , which is not strange in this class of solutions, while the exponent of one of the scale factors is irrational. This fact is odd since these constants usually are rational ones. Note that the solution is inflationary since $q < 0$ without any necessity to appeal to a DM component. In the third model, where G and Λ are considered as time-varying within the general relativity framework we have shown that the obtained solution is valid for all value of γ . In this case G may be a growing or decreasing function finding that it behaves as a true constant only when $\gamma = \gamma_c$. The dynamical cosmological constant is always a decreasing time function but it may be positive or negative and it vanishes if $\gamma = \gamma_c$. Once we know how each physical quantity works in the general relativity framework then we explore some solutions for the scalar-tensor models.

We have not been able to find a solution in the case of a Brans-Dicke model with a dynamical Λ . This result is quite

surprising since following the same procedure we have obtained solutions for several Bianchi models. Nevertheless for the BD model with a potential we obtain an exact self-similar solution. The numerical analysis carried out shows us that the solution obtained is only valid in a small neighbourhood, $\mathcal{E}(\gamma_c)$, of γ_c , the critical value of γ , where the energy density and the scalar function are positive. The solution shows analogies with that obtained in the subsection where G and Λ are considered as time-varying within the general relativity framework. For example, $G_{\text{eff}} \approx (t+t_0)^{-n}$ may be a growing or decreasing function finding that it behaves as a true constant only when $\gamma = \gamma_c$. The dynamical cosmological constant, the potential $U(\phi)$, is always a decreasing time function but it may be positive in the interval of definition or vanish if $\gamma = \gamma_c$. The model also accelerates since $q < 0$. Trying to generalize this scenario we also consider a chameleon BD model. We show that the obtained solution is valid for all values of γ instead of only in $\mathcal{E}(\gamma_c)$. In the last model studied, the induced gravity case, we find that the solution is only valid if $\gamma < -1$ (a phantom scenario). In the appendix we emphasize the fact that the only form compatible with self-similar solutions for the BD parameter $\omega_{BD}(\phi)$ is $\omega_{BD}(\phi) = \text{const}$. Therefore none of the scaling solutions admit a variable $\omega_{BD}(\phi)$.

A Study of Eq. (11)

In this appendix we shall study through the Lie group method the Eq. (11) i.e.

$$\phi_{tt} = -\frac{\phi_t^2}{\phi} \left(\frac{1}{2} \frac{\omega_\phi}{\omega} \phi - 1 \right) - \frac{\phi_t}{t}. \quad (130)$$

Therefore, following the standard procedure, we need to solve the next system of PDE:

$$W\xi_\phi - \phi\xi_{\phi\phi} = 0, \quad (131)$$

$$-W\eta + \frac{\phi}{2\omega} \left(\omega_{\phi\phi}\phi + \omega_\phi \left(1 - \frac{\omega_\phi}{\omega} \phi \right) \right) \eta + 2t^{-1}\phi^2\xi_\phi + W\phi\eta_\phi + \phi^2\eta_{\phi\phi} - 2\phi^2\xi_{t\phi} = 0, \quad (132)$$

$$t^{-2}(t\xi_t - \xi) + 2W\phi^{-1}\eta_t + 2\eta_{t\phi} - \xi_{tt} = 0, \quad (133)$$

$$t^{-1}\eta_t + \eta_{tt} = 0, \quad (134)$$

where $W = \frac{1}{2} \frac{\omega_\phi}{\omega} \phi - 1$.

The symmetry $\xi = t$, $\eta = n\phi$, brings us to obtain the following constrain on the function $\omega(\phi)$:

$$-W\eta + \frac{\phi}{2\omega} \left(\omega_{\phi\phi}\phi + \omega_\phi \left(1 - \frac{\omega_\phi}{\omega} \phi \right) \right) \eta + W\phi\eta_\phi = 0,$$

i.e.

$$-\left(\frac{1}{2} \frac{\omega_\phi}{\omega} \phi - 1 \right) n\phi + \frac{\phi}{2\omega} \left(\omega_{\phi\phi}\phi + \omega_\phi \left(1 - \frac{\omega_\phi}{\omega} \phi \right) \right) n\phi + \left(\frac{1}{2} \frac{\omega_\phi}{\omega} \phi - 1 \right) \phi n = 0,$$

and therefore we get

$$\omega_{\phi\phi}\phi + \omega_\phi \left(1 - \frac{\omega_\phi}{\omega} \phi \right) = 0,$$

so

$$\omega_{\phi\phi} = \frac{\omega_\phi^2}{\omega} - \frac{\omega_\phi}{\phi} \quad \Longleftrightarrow \quad \omega(\phi) = \omega_0\phi^\delta, \quad \omega_0, \delta \in \mathbb{R}.$$

Therefore if $\phi = \phi_0 t^n$, then $\omega(\phi) = \omega_0\phi^\delta = \omega_0 t^{n\delta}$.

If we substitute this result into Eq. (130) then we get

$$n(n-1) = -n^2 \left(\frac{1}{2} \delta - 1 \right) - n \quad \Longleftrightarrow \quad \delta = 0,$$

this means that

$$\omega(\phi) = \text{const.}$$

as we already know.

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